Distinguishing subgroups of the rationals by their Ramsey properties

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Abstract

A system of linear equations with integer coefficients is *partition regular* over a subset S of the reals if, whenever $S \setminus \{0\}$ is finitely coloured, there is a solution to the system contained in one colour class. It has been known for some time that there is an infinite system of linear equations that is partition regular over $\mathbb{R}$ but not over $\mathbb{Q}$, and it was recently shown (answering a long-standing open question) that one can also distinguish $\mathbb{Q}$ from $\mathbb{Z}$ in this way.

Our aim is to show that the transition from $\mathbb{Z}$ to $\mathbb{Q}$ is not sharp: there is an infinite chain of subgroups of $\mathbb{Q}$, each of which has a system that is partition regular over it but not over its predecessors. We actually prove something stronger: our main result is that if R and S are subrings of $\mathbb{Q}$ with R not contained in S, then there is a system that is partition regular over R but not over S. This implies, for example, that the chain above may be taken to be uncountable.

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1 Introduction

Consider a system of u linear equations in v unknowns.

$$\begin{array}{ccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + \cdots + & a_{1,v}x_v & = & 0 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + \cdots + & a_{2,v}x_v & = & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ a_{u,1}x_1 & + & a_{u,2}x_2 & + \cdots + & a_{u,v}x_v & = & 0 \end{array}$$

If the coefficients are rational numbers and the set $\mathbb{N}$ of positive integers is finitely coloured, can one be guaranteed to find monochromatic $x_1, x_2, \ldots, x_v$ solving the given system? That is, is the system of equations *partition regular*? In [8], Rado answered this question, showing that the system is partition regular if and only if the matrix of coefficients

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,v} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,v} \\ \vdots & \vdots & \ddots & \vdots \\ a_{u,1} & a_{u,2} & \cdots & a_{u,v} \end{pmatrix}$$

satisfies the *columns condition*:

Definition 1.1. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Z}$. Denote the columns of A by $\langle \vec{c}_i \rangle_{i=1}^{v}$. The matrix A satisfies the *columns condition* if there exist $m \in \{1, 2, \ldots, v\}$ and a partition $\langle I_t \rangle_{t=1}^{m}$ of $\{1, 2, \ldots, v\}$ such that

(1) $\sum_{i \in I_1} \vec{c}_i = \vec{0}$;

(2) for each $t \in \{2, 3, \ldots, m\}$, if any, $\sum_{i \in I_t} \vec{c}_i$ is a linear combination with coefficients from $\mathbb{Q}$ of $\{\vec{c}_i : i \in \bigcup_{j=1}^{t-1} I_j\}$.

If one considers the same equations over $\mathbb{R}$, an easy compactness argument shows that a finite system of equations is partition regular over the reals if and only if it is partition regular over the integers.

Note that the restriction to integer coefficients might as well be to rational coefficients, as we are always free to multiply each equation by a constant. We remark in passing that if one were to allow coefficients that are not rational, then the situation for finite systems is again understood: in [9], Rado extended his result by showing that if R is any subring of the set

$\mathbb{C}$ of complex numbers and the entries of A are from R, then the system of equations is partition regular over R if and only if the matrix A satisfies the columns condition over the field F generated by R (which means that we replace 'linear combination with coefficients from $\mathbb{Q}$' by 'linear combination with coefficients from F').

So the partition regularity of finite systems is quite settled. The case with *infinite* systems of linear equations, however, is much harder, and in general is still poorly understood. There is by now a large literature on the subject (see the survey [5]), but there is nothing resembling a characterisation of those infinite systems that are partition regular over $\mathbb{Z}$, $\mathbb{Q}$, or any other interesting subset of $\mathbb{C}$.

One difference between finite and infinite systems is the main focus of this paper. As stated above, if a finite system of linear equations has rational coefficients, it is a consequence of Rado's original theorems that the system is partition regular over $\mathbb{N}$ if and only if it is partition regular over $\mathbb{R}$ (and thus if and only if it is partition regular over $\mathbb{Z}$ or over $\mathbb{Q}$).

It was shown in [6] that the infinite system of equations $y_n = x_n - x_{n+1}$ ($n = 0, 1, 2, \ldots$) is partition regular over $\mathbb{R}$ but not over $\mathbb{Q}$. It was an open problem for some time whether every system of linear equations with rational coefficients that is partition regular over $\mathbb{Q}$ must also be partition regular over $\mathbb{N}$. (We remark in passing that there is no difference between $\mathbb{N}$ and $\mathbb{Z}$ in this regard, because if a system has a bad k-colouring over $\mathbb{N}$ then it also has a bad $2k$-colouring over $\mathbb{Z}$, obtained by copying the colouring of $\mathbb{N}$ to the negative integers but using k new colours—so we switch freely between $\mathbb{N}$ and $\mathbb{Z}$ in this paper.)

This question was answered in the negative in [1, Theorem 12] by showing that the following system of equations is partition regular over $\mathbb{D}$, the set of dyadic rationals. (It is not partition regular over $\mathbb{N}$ because it has no solutions in $\mathbb{N}$ at all.)

$$\begin{gathered}
x_{1,1} + 2^{-1}y = z_{1,1} \\
x_{2,1} + x_{2,2} + 2^{-2}y = z_{2,1} + z_{2,2} \\
\vdots \\
x_{n,1} + \cdots + x_{n,n} + 2^{-n}y = z_{n,1} + \cdots + z_{n,n} \\
\vdots
\end{gathered}$$

In this paper we extend this result by considering the following system of equations, which is a generalisation of another system introduced in [1].

Let $\alpha \in \mathbb{N}$ and, for $n \geq 2$ and $1 \leq i \leq \alpha$, let $d_{n,i}$ be an element of some infinite ring R. (We take rings to have identities.)

System $(*)$:

$$\begin{aligned} x_{2,1} + x_{2,2} + d_{2,1}y_1 + d_{2,2}y_2 + \cdots + d_{2,\alpha}y_\alpha &= z_2 \\ x_{3,1} + x_{3,2} + x_{3,3} + d_{3,1}y_1 + d_{3,2}y_2 + \cdots + d_{3,\alpha}y_\alpha &= z_3 \\ &\vdots \\ x_{n,1} + \cdots + x_{n,n} + d_{n,1}y_1 + d_{n,2}y_2 + \cdots + d_{n,\alpha}y_\alpha &= z_n \\ &\vdots \end{aligned}$$

In Section 3 we prove (Theorem 3.6) that, if R satisfies a certain technical condition, then System $(*)$ is partition regular over R. (This technical condition is satisfied by all subrings of $\mathbb{Q}$.) We actually show that System $(*)$ satisfies a slightly stronger condition: it is strongly partition regular over R.

Definition 1.2. Let R be a ring. A system of linear equations (with coefficients in R) is *strongly partition regular over* R if, whenever R is finitely coloured, there exists a monochromatic solution to the system with distinct variables taking on different values.

[This is the reason for starting System $(*)$ at $n = 2$. If we include the equation for $n = 1$, then the system remains partition regular, but we cannot ensure that $x_{1,1}$ and z_1 receive different colours: consider the case where $d_{1,1} = d_{1,2} = \cdots d_{1,\alpha} = 0$.]

In Section 4 we apply the results of Section 3 to show that there is an infinite increasing sequence $\langle G_n \rangle_{n=1}^{\infty}$ of subgroups of $\mathbb{Q}$ with the property that, for each n, there is a choice of the coefficients $\langle d_{n,i} \rangle_{n=1}^{\infty}$ making System $(*)$ strongly partition regular over G_{n+1} while it is not partition regular over G_n. We actually prove rather more (Theorem 4.3): this separation property holds for any two subrings of the rationals. This means that, for example, there is even an uncountable chain with this property. We close with some open problems.

The results of Section 3 make substantial use of the algebraic structure of the Stone–Čech compactification of a discrete semigroup, which we briefly introduce in Section 2.

2 The Stone–Čech compactification

Let S be a semigroup. We shall be concerned here exclusively with commutative semigroups, so we will denote the operation of S by $+$. For proofs of the assertions made here, see the first five chapters of [7].

The *Stone–Čech compactification* of S is denoted by βS. The points of βS are the ultrafilters on S. We identify the principal ultrafilters with the points of S, whereby we pretend that $S \subseteq \beta S$. The operation on S extends to an operation on βS, also denoted by $+$, with the property that, for $x \in S$ and $q \in \beta S$, the functions

$$\begin{aligned} p &\mapsto x + p \qquad \text{and} \\ q &\mapsto p + q \end{aligned}$$

are continuous. (The reader should be cautioned that $(\beta S, +)$ is almost certain to be non-commutative: the centre of $(\beta\mathbb{N}, +)$ is $\mathbb{N}$.) Given $A \subseteq S$ and $p, q \in \beta S$, $A \in p + q$ if and only if $\{x \in S : -x + A \in q\} \in p$, where $-x + A = \{y \in S : x + y \in A\}$.

With this operation, $(\beta S, +)$ is a compact Hausdorff right topological semigroup. Any such object contains idempotents, points p such that $p = p+p$. The semigroup βS has a smallest two-sided ideal, $K(\beta S)$, which is the union of all of the minimal right ideals of βS as well as the union of all of the minimal left ideals of βS. The intersection of any minimal right ideal with any minimal left ideal is a group (and any two such groups are isomorphic). In particular, there are idempotents in $K(\beta S)$—such idempotents are called *minimal*.

A subset A of S is

- an *IP-set* if it is a member of some idempotent;
- *central* if it is a member of some minimal idempotent;
- *central** if it is a member of every minimal idempotent;
- an *IP*-set* if it is a member of every idempotent.

Equivalently, A is an IP*-set if, whenever $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence in S, there exists $F \in \mathcal{P}_f(\mathbb{N})$ such that $\sum_{n \in F} x_n \in A$. We will use this to show that certain sets are central*.

We will also require the following more specialised results from [7].

Lemma 2.1. (a) *Let G be a commutative group. Then every minimal idempotent in βG is non-principal.*

(b) *Let S be a semigroup, let p be an idempotent in βS and, for $C \in p$, let $C^\star = \{s \in S : -s + C \in p\}$. Then $C^\star \in p$ and, for each $s \in C^\star$, we have $-s + C^\star \in p$.*

Proof. (a) Let p be a principal ultrafilter. Then p is idempotent if and only if $p+p=p$ in G, that is, if and only if $p=0$. Suppose that 0 were a minimal idempotent. Then by Theorem 1.48 of [7], $\beta S+0=\beta S$ is a minimal left ideal. But by Corollary 4.33 of [7], $\beta S\setminus S$ is a left ideal, contradicting the minimality of βS.

(b) Lemma 4.14 (and preceding discussion) of [7]. □

3 General results

In this section we will show that System (∗), with coefficients $d_{n,i}$ in some infinite ring R, is strongly partition regular over R. In fact, we shall establish a stronger conclusion.

Definition 3.1. Let $(S,+)$ be a semigroup.

- A system of linear equations is *centrally partition regular over* S if, whenever A is a central subset of S, there exists a solution to the system contained in A.
- A system of linear equations is *strongly centrally partition regular over* S if, whenever A is a central subset of S, there exists a solution to the system contained in A with distinct variables taking on different values.

Notice that, since whenever a semigroup is finitely coloured, one colour class must be central, it follows that, if a system of equations is strongly centrally partition regular, then it is strongly partition regular.

We use the usual additive notation

$$\begin{aligned} A+B&=\{a+b: a\in A, b\in B\}\\ A-B&=\{a-b: a\in A, b\in B\}\\ kA&=A+\cdots+A \quad (k \text{ times})\end{aligned}$$

and write $k\cdot A=\{k\cdot a: a\in A\}$.

We shall need the following result from [2].

Lemma 3.2. *Let $(G,+)$ be a commutative group and assume that $c\cdot G$ is a central* set for each $c\in\mathbb{N}$. Let C be a central subset of G. Then there is an $m\in\mathbb{N}$ and a k such that, if $n\geq k$, then $m\cdot G\subseteq C-nC$.*

Proof. [2, Lemma 3.7]. □

Definition 3.3. Let A be a $u \times v$ matrix with entries from a ring R. An element $a_{i,j}$ of A is a *first entry* of A if $a_{i,k} = 0$ for $k < j$ and $a_{i,j} \neq 0$. We say that A satisfies the *weak first entries condition* if no row of A is $\vec{0}$ and if $a_{i,k}$ and $a_{j,k}$ are first entries of A, then $a_{i,k} = a_{j,k}$.

We call this the *weak* first entries condition because, as usually defined with $R = \mathbb{Q}$, one assumes that first entries are positive—which of course does not make sense for general rings.

Lemma 3.4. *Let R be an infinite ring. Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from R that satisfies the weak first entries condition, and suppose that $c \cdot R$ is central* in R for each first entry c of A. Let C be central in R. Then there is an $\vec{x}$ in $(R \setminus \{0\})^v$ such that $A\vec{x} \in C^u$.*

Proof. This is a special case of [7, Theorem 15.5]. That theorem was stated only for coefficients which were natural numbers so that it made sense in an arbitrary semigroup, but the proof in the case of rings is nearly identical. □

Theorem 3.5. *Let R be an infinite ring and assume that, for each $m \in \mathbb{N}$, $m \cdot R$ is central* in R. Let $\alpha \in \mathbb{N}$ and, for each $n \geq 2$ and $1 \leq i \leq \alpha$, let $d_{n,i}$ be in R. Then for each central subset C of R there is a solution*

$$y_1, y_2, \ldots, y_\alpha, x_{2,1}, x_{2,2}, z_2, x_{3,1}, x_{3,2}, x_{3,3}, z_3, \ldots$$

of System $(*)$ contained in C; that is, System $(*)$ is centrally partition regular over R. Moreover, the solution can be chosen so that y_1, y_2, $\ldots$, y_α are distinct.

Proof. Let C be central in R. There is an idempotent $p \in \beta S$ such that $C \in p$, and by Lemma 2.1(a), $p \neq 0$. Hence $C \setminus \{0\} \in p$, so $C \setminus \{0\}$ is also central and we may assume that $0 \notin C$.

By Lemma 3.2, there is an $m \in \mathbb{N}$ and a k such that, if $n \geq k$, then $m \cdot G \subseteq C - nC$. Since $m \cdot G$ is central*, $C \cap m \cdot G$ is central.

Let $b_1 = 0$ and, for $2 \leq j \leq k$, let $b_j = b_{j-1} + j$. Let $v = b_k + \alpha$ and let A be the $(k-1) \times v$ matrix with entries given by

$$a_{i,j} = \begin{cases} 1 & \text{if } b_i < j \leq b_{i+1}; \\ d_{i+1,t} & \text{if } j = b_k + t; \\ 0 & \text{otherwise.} \end{cases}$$

Let B be an $\binom{\alpha}{2} \times v$ matrix such that for every $b_k < i < j \leq b_k + \alpha$, some row of B has a 1 in position i and a -1 in position j, with all other entries equal to 0. (If $\alpha = 1$, let B be empty.) Thus, for example, if $k = 4$ and $\alpha = 3$, then

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{2,1} & d_{2,2} & d_{2,3} \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & d_{3,1} & d_{3,2} & d_{3,3} \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & d_{4,1} & d_{4,2} & d_{4,3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Let I be the $v \times v$ identity matrix. Then $\begin{pmatrix} I \\ A \\ B \end{pmatrix}$ satisfies the first entries condition with each first entry equal to 1, so by Theorem 3.4 there exist

$$x_{2,1}, x_{2,2}, x_{3,1}, x_{3,2}, x_{3,3}, \ldots, x_{k,1}, x_{k,2}, \ldots, x_{k,k}, y_1, y_2, \ldots, y_\alpha$$

such that all entries of

$$\begin{pmatrix} I \\ A \\ B \end{pmatrix} \begin{pmatrix} x_{2,1} \\ \vdots \\ x_{k,k} \\ y_1 \\ \vdots \\ y_\alpha \end{pmatrix}$$

are in $C \cap m \cdot G$.

For $2 \leq n \leq k$, let $z_n = x_{n,1} + \cdots x_{n,n} + d_{n,1}y_1 + \cdots + d_{n,\alpha}y_\alpha$. The submatrix I ensures that the $x_{n,j}$ $(2 \leq n \leq k$ and $1 \leq j \leq n)$ and y_i $(1 \leq i \leq \alpha)$ are in C, the submatrix A ensures that the z_n $(2 \leq n \leq k)$ are in C, and the submatrix B ensures that $y_i \neq y_j$ $(1 \leq i < j \leq \alpha)$ as $y_i - y_j \in C \subseteq R \setminus \{0\}$.

For $n > k$, $d_{n,1}y_1 + \cdots + d_{n,\alpha}y_\alpha \in m \cdot G \subseteq C - nC$, so choose z_n and $x_{n,1} \ldots x_{n,n}$ in C such that $d_{n,1}y_1 + \cdots + d_{n,\alpha}y_\alpha = z_n - x_{n,1} - \cdots - x_{n,n}$. □

We are now ready for the main result of this section.

Theorem 3.6. *Let R be an infinite ring and assume that, for each $m \in \mathbb{N}$, $m \cdot R$ is central* in R. Let $\alpha \in \mathbb{N}$ and, for each $n \geq 2$ and $1 \leq i \leq \alpha$, let $d_{n,i}$ be in R. Then System $(*)$ is strongly centrally partition regular over R.*

Proof. We already know that System $(*)$ is centrally partition regular, by Theorem 3.5. We will apply Theorem 3.5 to a different central set ($C^\star$, defined below), then use that solution to build a solution in C with all values of the variables distinct. This will be possible because at each stage we will only have finitely many previously used values to avoid: since the minimal idempotent p witnessing the fact that various sets X are central is non-principal, sets obtained from X by deleting finitely many elements remain in p.

So let C be a central subset of G and pick a minimal idempotent $p \in \beta G$ such that $C \in p$. As in the proof of Theorem 3.5, p is non-principal and we can assume that $0 \notin C$. For each $B \in p$, let $B^\star = \{x \in B : B - x \in p\}$. If $B \in p$ and $x \in B^\star$, then by Lemma 2.1(b), $B^\star - x \in p$.

Again by Lemma 2.1(b), we have that $C^\star$ is central, so pick by Theorem 3.5 a solution

$$y_1, y_2, \dots, y_\alpha, x_{2,1}, x_{2,2}, z_2, x_{3,1}, x_{3,2}, x_{3,3}, z_3, \dots$$

of System $(*)$ contained in $C^\star$ such that the values of the y_i are distinct. We will use this solution to build a new solution in variables y_i, $u_{i,j}$ and v_i for which the values taken by the variables are all distinct.

Suppose that, for $2 \leq i < n$ and $1 \leq j \leq i$, we have already chosen $u_{i,j}$ and v_i distinct from each other and from $y_1, y_2, \dots, y_\alpha$ such that

$$u_{i,1} + \cdots + u_{i,i} + d_{i,1}y_1 + \cdots + d_{i,\alpha}y_\alpha = v_i.$$

We will choose $w_1, \dots, w_n$ in such a way that, setting $u_{n,i} = x_{n,i} + w_i$ and $v_n = z_n + w_1 + \cdots + w_n$, the same is true with n replaced by $n+1$.

Let

$$\begin{aligned} A &= (C - x_{n,1}) \cap (C - x_{n,2}) \cap \cdots \cap (C - x_{n,n}) \cap (C - z_n), \\ B &= \{y_1, \dots y_\alpha\} \cup \{u_{i,j} : 2 \leq i < n, 1 \leq j \leq i\} \\ &\qquad \cup \{v_i : 2 \leq i < n\} \cup \{x_{n,i} - z_n : 1 \leq i \leq n\}, \text{ and} \\ D &= A \setminus (B \cup (B - x_{n,1}) \cup \cdots \cup (B - x_{n,n}) \cup (B - z_n)). \end{aligned}$$

Since the $x_{i,j}$ and z_i are in $C^\star$, $A \in p$. Since B is finite, $D \in p$. Choose $w_1 \in D^\star$.

Let $2 \leq k \leq n$ and suppose that we have already chosen $w_1, \dots, w_{k-1}$ such that

(i) if $\emptyset \neq F \subseteq \{1, 2, \dots, k-1\}$, then $\sum_{j \in F} w_j \in D^\star$, and

(ii) if $1 \leq i < j \leq k-1$, then $x_{n,i} + w_i \neq x_{n,j} + w_j$.

Choose

$$w_k \in D^\star \cap \bigcap_{\emptyset \neq F \subseteq \{1,2,\ldots,k-1\}} (D^\star - \sum_{j\in F} w_j) \setminus \{x_{n,j} + w_j - x_{n,k} : 1 \leq j < k\}.$$

Then (i) and (ii) hold with k replaced by $k+1$.

Having chosen $w_1, \ldots, w_n$, let $u_{n,i} = x_{n,i} + w_i$ and let $v_n = z_n + w_1 + \cdots + w_n$. By (i), $w_1, \ldots, w_n$ and $w_1 + \cdots + w_n$ are each in $D^\star \subseteq D$. Hence by the definition of A, $u_{n,1}, \ldots, u_{n,n}$ and v_n are all in C, and by the definitions of B and D, $u_{n,1}, \ldots, u_{n,n}$ and v_n are all distinct from the y_i $(1 \leq i \leq \alpha)$, $u_{i,j}$ $(2 \leq i < n$ and $1 \leq j \leq i)$ and v_i $(2 \leq i < n)$. By (ii), the $u_{n,j}$ are all distinct. Finally, suppose that $v_n = u_{n,j}$ for some j. Then $w_1 + \cdots + w_{j-1} + w_{j+1} + \cdots + w_n = x_{n,j} - z_n \in B$, but by (i) $w_1 + \cdots + w_{j-1} + w_{j+1} + \cdots + w_n \in D^\star \subseteq D$, which is a contradiction. □

4 Applications

In this section we show that for any two subrings R and S of $\mathbb{Q}$ such that R is not contained in S, there is a system that is partition regular over R but not over S. In fact, we shall obtain this for a choice of the sequence $\langle d_{n,1} \rangle_{n=1}^\infty$, making System $(*)$ strongly centrally partition regular over R while it has no solutions in S.

In particular, this will give us a chain of $\mathfrak{c}$ subgroups of $\mathbb{Q}$ (where $\mathfrak{c}$ is the cardinality of the continuum), any two of which have different partition regular systems, as stated in the introduction. (To see that any countable set has a chain of subsets ordered by $\mathbb{R}$, simply consider $\big\{\{x \in \mathbb{Q} : x < y\} : y \in \mathbb{R}\big\}$.)

Definition 4.1. Let P be the set of primes and let $F \subseteq P$. Then

$$\mathbb{G}_F = \{a/b : a \in \mathbb{Z}, b \in \mathbb{N} \text{ and all prime factors of } b \text{ are in } F\}.$$

Thus $\mathbb{G}_\emptyset = \mathbb{Z}$, $\mathbb{G}_{\{2\}} = \mathbb{D}$ and $\mathbb{G}_P = \mathbb{Q}$. It is easy to check that the $\mathbb{G}_F$ are precisely the subrings of $\mathbb{Q}$. (Given a subring R of $\mathbb{Q}$, let $F = \{p \in P : \frac{1}{p} \in R\}$ and use the fact that $1 \in R$.)

We will invoke Theorem 3.6, so we need to know that for any subset F of P and any $m \in \mathbb{N}$, $m \cdot \mathbb{G}_F$ is central* in $\mathbb{G}_F$. We will in fact show that it is IP*. Recall that this means that, given any sequence $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{G}_F$, there is some $H \in \mathcal{P}_f(\mathbb{N})$ such that $\sum_{n\in H} x_n \in m \cdot \mathbb{G}_F$ or, equivalently, that $m \cdot \mathbb{G}_F$ is a member of every idempotent in $\beta\mathbb{G}_F$.

Proposition 4.2. *Let $m \in \mathbb{N}$, $F \subseteq P$ and $\langle x_n \rangle_{n=1}^{(m-1)^2+1}$ be a sequence of elements of $\mathbb{G}_F$. Then there exists $\emptyset \neq H \subseteq \{1, 2, \ldots, (m-1)^2 + 1\}$ such that $\sum_{n \in H} x_n \in m \cdot \mathbb{G}_F$.*

Proof. Write the x_n over a common denominator: choose $s \in \mathbb{N}$ with all prime factors in F such that $x_n = y_n/s$ with $y_n \in \mathbb{Z}$. At least m of the y_n must have the same residue modulo m; let H be a set of size m such that $y_n \equiv h \pmod{m}$ for $n \in H$. Then $\sum_{n \in H} y_n = km$ for some k, hence $\sum_{n \in H} x_n = km/s \in m \cdot \mathbb{G}_F$. □

Theorem 4.3. *Let F and H be subsets of P with $H \setminus F \neq \emptyset$ and pick $q \in H \setminus F$. Let $\alpha = 1$ and for $k \in \mathbb{N}$, let $d_{n,1} = \frac{1}{q^n}$. Then System $(*)$ is strongly centrally partition regular over $\mathbb{G}_H$ but is not partition regular over $\mathbb{G}_F$.*

Proof. It is immediate that System $(*)$ has no solutions in $\mathbb{G}_F$. By Theorem 3.6 with $R = \mathbb{G}_H$, System $(*)$ is strongly centrally partition regular over $\mathbb{G}_H$. □

By applying this to a chain of size $\mathfrak{c}$ of subsets of the primes, we immediately obtain a chain of $\mathfrak{c}$ subrings of $\mathbb{Q}$, no two of which have the same partition regular systems.

If we want to separate $\mathbb{Q}$ from all proper subrings simultaneously then we have the following, whose proof is identical. Let $p_1, p_2, \ldots$ be an enumeration of the primes.

Theorem 4.4. *Let $\alpha = 1$ and for $n \in \mathbb{N}$, let $d_{n,1} = \prod_{t=1}^{n} \frac{1}{p_t^n}$. Then System $(*)$ is strongly centrally partition regular over $\mathbb{Q}$, but is not partition regular over $\mathbb{G}_F$ for any proper subset F of P.* □

One might raise the objection that it almost seems like cheating to show that a system is not partition regular over G by showing that it has no solutions there at all. We see now that by taking $\alpha = 2$, we can get examples where System $(*)$ has solutions in $\mathbb{N}$, but the conclusions of Theorems 4.3 and 4.4 still hold.

Theorem 4.5. *Let F and H be subsets of P with $H \setminus F \neq \emptyset$ and pick $q \in H \setminus F$. Let $\alpha = 2$ and, for $n \in \mathbb{N}$, let $d_{n,1} = \frac{-1}{q^n}$ and $d_{n,2} = \frac{2}{q^n}$. Then System $(*)$ has solutions in $\mathbb{N}$ and is strongly centrally partition regular over $\mathbb{G}_H$, but is not partition regular over $\mathbb{G}_F$.*

Proof. By Theorem 3.6 with $R = \mathbb{G}_H$, System $(*)$ is strongly centrally partition regular over $\mathbb{G}_H$. Let $y_1 = 2$ and $y_2 = 1$. Then for every $n \in \mathbb{N}$, $d_{n,1}y_1 + d_{n,2}y_2 = 0$ so it is easy to find a solution to System $(*)$ in $\mathbb{N}$.

To see that System $(*)$ is not partition regular over $\mathbb{G}_F$, two-colour $\mathbb{G}_F \setminus \{0\}$ so that for all $x \in \mathbb{G}_F \setminus \{0\}$, x and $2x$ do not have the same colour. (For example colour by the parity of $\lfloor \log_2(|x|) \rfloor$.) Suppose we have a monochromatic solution to System $(*)$ in $\mathbb{G}_F$. We have that $y_1 \neq 2y_2$ and, for each $n \in \mathbb{N}$, $(2y_2 - y_1)/q^n = z_n - x_{n,1} - \cdots - x_{n,n} \in \mathbb{G}_F$, which is a contradiction for n sufficiently large. $\square$

Similarly, we have an analogue of Theorem 4.4.

Theorem 4.6. *Let $\alpha = 2$ and for $n \in \mathbb{N}$, let $d_{n,1} = \prod_{t=1}^{n} \frac{-1}{p_t^n}$ and $d_{n,2} = \prod_{t=1}^{n} \frac{2}{p_t^n}$. Then System $(*)$ is strongly partition regular over $\mathbb{Q}$ and has solutions in $\mathbb{N}$, but is not partition regular over $\mathbb{G}_F$ for any proper nonempty subset F of P.* $\square$

Let us end by remarking that it would be interesting to understand what happens beyond $\mathbb{Q}$—in other words, for subrings (or subgroups) that lie between $\mathbb{Q}$ and $\mathbb{R}$. Of course, if one allows non-rational coefficients then it is easy to separate sets, so the interest would be for systems of equations whose coefficients are integers (or, equivalently, rationals).

We see now that the system mentioned in the Introduction that distinguishes $\mathbb{R}$ from $\mathbb{Q}$ in fact distinguishes any uncountable subgroup G of $\mathbb{R}$ from $\mathbb{Q}$.

In the following result we use, as in [6], the Baumgartner–Hajnal theorem [3, Theorem 1]. This theorem states that if A is a linearly ordered set with the property that whenever $\varphi : A \to \mathbb{N}$, there is an infinite increasing sequence in A on which φ is constant, then for any countable ordinal α, and any finite colouring ψ of the two-element subsets of A there is a subset B of A which has order type α such that ψ is constant on the two-element subsets of B. (The theorem was proved in [3] using Martin's axiom followed by an absoluteness argument to show that it is a theorem of ZFC. A direct combinatorial proof was obtained by Galvin in [4, Theorem 4].)

Theorem 4.7. *Let G be an uncountable subgroup of $\mathbb{R}$. Then the system of equations $y_n = x_n - x_{n+1}$ $(n = 0, 1, 2, \ldots)$ is partition regular over G but not over $\mathbb{Q}$.*

Proof. It was shown in [6] (immediately before Question 6) that the system is not partition regular over $\mathbb{Q}$. To show that the system is partition regular over G, we use the Baumgartner–Hajnal theorem. For this we need to observe that given any countable colouring of G, there is a monochromatic increasing sequence. To see this, let $\varphi : G \to \mathbb{N}$ and define $\psi : G \to \mathbb{N} \times \mathbb{N}$ by $\psi(x) = \big(\varphi(x), \varphi(-x)\big)$. Pick $(n, m) \in \mathbb{N} \times \mathbb{N}$ such that $A = \psi^{-1}[\{(n, m)\}]$ is infinite. Then A contains a sequence $\langle x_t \rangle_{t=1}^{\infty}$ which is either increasing or decreasing. If $\langle x_t \rangle_{t=1}^{\infty}$ is increasing, then it is an increasing sequence in $\varphi^{-1}[\{n\}]$. If $\langle x_t \rangle_{t=1}^{\infty}$ is decreasing, then $\langle -x_t \rangle_{t=1}^{\infty}$ is an increasing sequence in $\varphi^{-1}[\{m\}]$.

Now let G be finitely coloured by φ and, given a two-element subset $\{x, y\}$ of G, define $\psi(\{x, y\}) = \varphi(|x - y|)$. By the Baumgartner–Hajnal theorem, pick an increasing sequence $\langle z_\sigma \rangle_{\sigma < \omega+1}$ such that ψ is constant on $\big\{\{z_\sigma, z_\tau\} : \sigma < \tau\big\}$. Given $n \in \mathbb{N}$, let $x_n = z_\omega - z_n$ and let $y_n = z_{n+1} - z_n$. □

Perhaps even more interesting would be to understand what happens for subgroups of $\mathbb{Q}$. The following is the obvious question to ask.

Question 4.8. *If G and H are subgroups of $\mathbb{Q}$ such that G does not contain a subgroup isomorphic to H, must there exist a system (of linear equations with integer coefficients) that is partition regular over H but not over G?*

It is easy to check that every subgroup of $\mathbb{Q}$ that contains 1 is the set of rationals a/b such that the multiplicity of p_i in the prime factorisation of b is at most k_i, where each k_i is either a non-negative integer or ∞. Given two such sequences k and k', if there is some i for which $k_i = \infty$ while k'_i is finite, then the corresponding groups can be separated by the methods of this section. But if for every i, either both k_i and k'_i are infinite, or both are finite, then we are unable to say anything.

The most attractive special case is surely the following.

Question 4.9. *Does there exist a system (of linear equations with integer coefficients) that is partition regular over the set of rationals with squarefree denominators but is not partition regular over the integers?*

References

[1] B. Barber, N. Hindman, and I. Leader, *Partition regularity in the rationals*, J. Comb. Theory (Series A) **120** (2013), 1590–1599.

[2] B. Barber, N. Hindman, I. Leader, and D. Strauss, *Partition regularity without the columns property* , Proc. Amer. Math. Soc., to appear[1].

[1]Currently available at `http://nhindman.us/preprint.html`.

[3] J. Baumgartner and A. Hajnal, *A proof (involving Martin's Axiom) of a partition relation*, Fund. Math. **78** (1973), 193–203.

[4] F. Galvin, *On a partition theorem of Baumgartner and Hajnal*, Colloq. Math. Soc. János Bolyai, Vol. 10, North Holland, Amsterdam, 1975, 711-729.

[5] N. Hindman, *Partition regularity of matrices*, Integers **7(2)** (2007), A-18. `http://www.integers-ejcnt.org/vol7-2.html`

[6] N. Hindman, I. Leader, and D. Strauss, *Open problems in partition regularity*, Combin. Probab. Comput. **12** (2003), 571–583.

[7] N. Hindman and D. Strauss, *Algebra in the Stone–Čech compactification: theory and applications, 2nd edition*, Walter de Gruyter & Co., Berlin, 2012.

[8] R. Rado, *Studien zur Kombinatorik*, Math. Z. **36** (1933), 424–470.

[9] R. Rado, *Note on combinatorial analysis*, Proc. London Math. Soc. **48** (1943), 122–160.